

Proof of some conjectures of Z.-W. Sun on the divisibility of certain double-sums

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Abstract. Z.-W. Sun introduced three kinds of numbers:

$$S_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} (2k+1), \quad s_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \frac{1}{2k-1},$$

and $S_n^+ = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} (2k+1)^2$. In this paper we mainly prove that

$$4 \sum_{k=0}^{n-1} k S_k \equiv \sum_{k=0}^{n-1} s_k \equiv \sum_{k=0}^{n-1} S_k^+ \equiv 0 \pmod{n^2} \quad \text{for } n \geq 1,$$

by establishing some binomial coefficient identities, such as

$$4 \sum_{k=0}^{n-1} k S_k = n^2 \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{2k}{k} \left(6k \binom{n-1}{k}^2 + \binom{n-1}{k} \binom{n-1}{k+1} \right).$$

This confirms several recent conjectures of Z.-W. Sun.

Keywords: congruences; Legendre symbol; Zeilberger's algorithm;

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1 Introduction

Inspired by the Schröder numbers in combinatorics, Z.-W. Sun [4] introduced the following two kinds of numbers

$$R_n = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} \frac{1}{2k-1}, \quad S_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} (2k+1),$$

and proved many remarkable arithmetic properties of these numbers. For example, he proved that, for any odd prime p and positive integer n ,

$$\begin{aligned} \sum_{k=0}^{p-1} R_k &\equiv -p - \left(\frac{-1}{p} \right) \pmod{p^2}, \\ \frac{1}{n^2} \sum_{k=0}^{n-1} S_k &= \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{2k}{k} \frac{1}{k+1}, \end{aligned} \tag{1.1}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

The aim of this paper is to prove the following results.

Theorem 1.1 *Let n be a positive integer and p a prime. Then*

$$4 \sum_{k=0}^{n-1} k S_k \equiv 0 \pmod{n^2}, \quad (1.2)$$

$$\sum_{k=0}^{p-1} k S_k \equiv \frac{p^2}{8} \left(5 - 9 \left(\frac{p}{3}\right)\right) \pmod{p^3}. \quad (1.3)$$

Let

$$s_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \frac{1}{2k-1}, \quad \text{and} \quad S_n^+ = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} (2k+1)^2.$$

Theorem 1.2 *Let n be a positive integer and p a prime. Then*

$$\sum_{k=0}^{n-1} s_k \equiv 0 \pmod{n^2}, \quad (1.4)$$

$$\sum_{k=0}^{p-1} s_k \equiv -\frac{p^2}{2} \left(1 + 9 \left(\frac{p}{3}\right)\right) \pmod{p^3}, \quad (1.5)$$

Theorem 1.3 *Let n be a positive integer and p a prime. Then*

$$\sum_{k=0}^{n-1} S_k^+ \equiv 0 \pmod{n^2}, \quad (1.6)$$

$$\sum_{k=0}^{p-1} S_k^+ \equiv -p^2 \left(\frac{p}{3}\right) \pmod{p^3}. \quad (1.7)$$

The congruences (1.2)–(1.4), and (1.6) were originally conjectured by Z.-W. Sun [4, Conjectures 5.5 and 5.6]. Note that, by establishing a lemma on sums of q -binomial coefficients, Z.-W. Sun himself could prove the congruences (1.4) and (1.6) modulo n .

The key idea in this paper is to find out new expressions of the sums $\frac{4}{n^2} \sum_{k=0}^{n-1} k S_k$, $\frac{1}{n^2} \sum_{k=0}^{n-1} s_k$ and $\frac{1}{n^2} \sum_{k=0}^{n-1} S_k^+$, just like Sun's identity (1.1). No doubt that Zeilberger's algorithm (see [1, 3]) will play an important role in our proof of Theorems 1.1–1.3. We shall give two proofs of (1.4), one of which also gives the following new congruence:

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{2k+1}{k} \frac{3}{4k^2-1} \equiv 0 \pmod{4n-1}. \quad (1.8)$$

2 Proof of Theorem 1.1

Proof of (1.2). It is well known that $\frac{1}{k+1}\binom{2k}{k}$ is an integer (the k th Catalan number). We shall prove the congruence (1.2) by establishing the following identity:

$$4 \sum_{k=0}^{n-1} k S_k = n^2 f_{n-1}, \quad (2.1)$$

where

$$f_n = \sum_{k=0}^n \frac{1}{k+1} \binom{2k}{k} \left(6k \binom{n}{k}^2 + \binom{n}{k} \binom{n}{k+1} \right).$$

It is clear that (2.1) holds for $n = 1$. It remains to show that

$$4n S_n = (n+1)^2 f_n - n^2 f_{n-1}, \quad (2.2)$$

for all positive integers n . Define $u_n = 4n S_n$. Applying Zeilberger's algorithm, we find that the numbers u_n satisfy the following recurrence:

$$\begin{aligned} & n(n+1)(n+2)(n+3)u_{n+3} - n(n+1)(n+3)(11n+29)u_{n+2} \\ & + n(n+2)(19n^2 + 74n + 87)u_{n+1} - 9(n+1)^3(n+2)u_n = 0. \end{aligned} \quad (2.3)$$

It is interesting that we can deduce the following second-order recurrence for u_n from (2.3):

$$\begin{aligned} & n(n+1)(n+2)(4n+3)(4n+7)u_{n+2} - n(4n+3)(4n+11)(10n^2 + 30n + 23)u_{n+1} \\ & + 9(n+1)^3(4n+11)(4n+7)u_n = 0. \end{aligned} \quad (2.4)$$

In fact, if we denote the left-hand sides of (2.3) and (2.4) by α_n and β_n , respectively, then we can easily check that

$$(4n+11)(4n+7)\alpha_n + n\beta_{n+1} - (n+2)\beta_n = 0. \quad (2.5)$$

Therefore, by induction on n , we immediately get $\beta_n = 0$, i.e., the recurrence (2.4) is true.

Let $v_n = (n+1)^2 f_n - n^2 f_{n-1}$. Then Zeilberger's algorithm gives the following relation:

$$\begin{aligned} & (n+2)(n+3)(128n^4 + 864n^3 + 2016n^2 + 1994n + 693)v_{n+3} \\ & - (1408n^6 + 17696n^5 + 88512n^4 + 225582n^3 + 309049n^2 + 215886n + 59535)v_{n+2} \\ & + (2432n^6 + 30880n^5 + 155712n^4 + 399646n^3 + 550013n^2 + 384657n + 106920)v_{n+1} \\ & - 9(n+1)^2(128n^4 + 1376n^3 + 5376n^2 + 9130n + 5695)v_n = 0. \end{aligned} \quad (2.6)$$

Similarly as before, we can deduce the following simpler recurrence for v_n from (2.6):

$$\begin{aligned} & n(n+1)(n+2)(4n+3)(4n+7)v_{n+2} - n(4n+3)(4n+11)(10n^2 + 30n + 23)v_{n+1} \\ & + 9(n+1)^3(4n+11)(4n+7)v_n = 0, \end{aligned} \quad (2.7)$$

by noticing that

$$(4n+11)(4n+7)(n+1)\gamma_n + (128n^4 + 864n^3 + 2016n^2 + 1994n + 693)\delta_{n+1} - (128n^4 + 1376n^3 + 5376n^2 + 9130n + 5695)\delta_n = 0,$$

where γ_n and δ_n denote the left-hand sides of (2.6) and (2.7), respectively.

Thus, we have proved that the sequences u_n and v_n satisfy the same recurrence (2.4) (i.e., (2.7)). Also, it is easy to verify that $u_n = v_n$ for $n = 1, 2$. This proves that $u_n = v_n$ for all positive integers. Namely, the identity (2.2) is true. This completes the proof. \square

Proof of (1.3). Letting $n = p$ be a prime in (2.1), we obtain

$$\sum_{k=0}^{p-1} kS_k = \frac{p^2}{4} \sum_{k=0}^{p-1} \frac{1}{k+1} \binom{2k}{k} \left(6k \binom{p-1}{k}^2 + \binom{p-1}{k} \binom{p-1}{k+1} \right). \quad (2.8)$$

It is easy to see that $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$ for $0 \leq k \leq p-1$. Therefore, from (2.8) we deduce that

$$\sum_{k=0}^{p-1} kS_k \equiv \frac{p^2}{4} \left(\sum_{k=0}^{p-1} \frac{6k}{k+1} \binom{2k}{k} - \sum_{k=0}^{p-2} \frac{1}{k+1} \binom{2k}{k} \right) \pmod{p^3}.$$

The proof then follows from the following two congruences due to Pan and Sun [2, (1.4) and (1.16) with $a = 0$] (see also [5]):

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3} \right) \pmod{p}, \quad (2.9)$$

$$\sum_{k=0}^{p-1} \frac{1}{k+1} \binom{2k}{k} \equiv \frac{1}{2} \left(3 \left(\frac{p}{3} \right) - 1 \right) \pmod{p}, \quad (2.10)$$

and the fact that

$$\frac{1}{p} \binom{2p-2}{p-1} \equiv -1 \pmod{p}. \quad (2.11)$$

3 Proof of Theorem 1.2

We shall give two proofs of (1.4) by establishing two different identities.

First proof of (1.4). We want to show that

$$\sum_{k=0}^{n-1} s_k = n^2 g_{n-1}, \quad (3.1)$$

where

$$g_n = \sum_{k=0}^n \frac{1}{k+1} \binom{2k}{k} \left(2 \binom{n}{k} \binom{n}{k+1} - \binom{n}{k}^2 \right).$$

Let $w_n = (n+1)^2 g_n - n^2 g_{n-1}$. Applying Zeilberger's algorithm, we find that s_n and w_n satisfy the same recurrence:

$$(n+3)^2 s_{n+3} - (11n^2 + 46n + 47) s_{n+2} + (19n^2 + 58n + 63) s_{n+1} - 9(n+1)^2 s_n = 0. \quad (3.2)$$

Moreover, it is easy check that $s_n = w_n$ for $n = 1, 2, 3$. This proves that $s_n = w_n$ for all positive integers n . The identity (3.1) then follows from the fact that $s_0 = g_0 = -1$. \square

Second proof of (1.4). We shall prove that

$$\sum_{k=0}^{n-1} s_k = \frac{n^2}{4n-1} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{2k+1}{k} \frac{3}{4k^2-1}. \quad (3.3)$$

We claim that the numbers s_n satisfy a second-order recurrence:

$$(n+2)^2 (8n+5) s_{n+2} - (80n^3 + 234n^2 + 214n + 63) s_{n+1} + 9(n+1)^2 (8n+13) s_n = 0. \quad (3.4)$$

Denote the left-hand sides of (3.2) and (3.4) by λ_n and μ_n , respectively. Then it is easy to verify that

$$(8n+13) \lambda_n + \mu_{n+1} - \mu_n = 0,$$

which leads to (3.4) by induction on n .

Let

$$h_n = \frac{1}{4n+3} \sum_{k=0}^n \binom{n}{k}^2 \binom{2k+1}{k} \frac{3}{4k^2-1},$$

and $x_n = (n+1)^2 h_n - n^2 h_{n-1}$. Applying Zeilberger's algorithm for x_n , we obtain

$$\begin{aligned} & (n+3)^2 (4n+15) (512n^3 + 2752n^2 + 4504n + 2069) x_{n+3} - (22528n^6 + 297728n^5 \\ & + 1584608n^4 + 4336564n^3 + 6403785n^2 + 4793214n + 1395765) x_{n+2} \\ & + (38912n^6 + 453376n^5 + 2138848n^4 + 5339252n^3 + 7595337n^2 + 5794002n \\ & + 1757133) x_{n+1} - 9(n+1)^2 (4n-1) (512n^3 + 4288n^2 + 11544n + 9837) x_n = 0. \end{aligned} \quad (3.5)$$

Similarly as before, the numbers x_n also satisfy a simpler recurrence:

$$(n+2)^2 (8n+5) x_{n+2} - (80n^3 + 234n^2 + 214n + 63) x_{n+1} + 9(n+1)^2 (8n+13) x_n = 0, \quad (3.6)$$

since

$$\begin{aligned} & (8n+13) \nu_n - (4n+15) (512n^3 + 2752n^2 + 4504n + 2069) \tau_{n+1} \\ & + (4n-1) (512n^3 + 4288n^2 + 11544n + 9837) \tau_n = 0, \end{aligned}$$

where ν_n and τ_n denote the left-hand sides of (3.5) and (3.6), respectively.

Thus, we have proved that s_n and x_n satisfy the same recurrence (3.4) (i.e., (3.6)). The proof of (3.3) then follows from the fact that $s_1 = x_1$ and $s_2 = x_2$.

Note that $\gcd(n^2, 4n - 1) = 1$ and

$$\binom{2k+1}{k} \frac{3}{4k^2-1} = \binom{2k}{k} \left(\frac{2}{2k-1} - \frac{1}{k+1} \right)$$

is an integer. The identity (3.3) immediately implies that both (1.4) and (1.8) hold. \square

Proof of (1.5). Letting $n = p$ be a prime in (3.1), we get

$$\begin{aligned} \sum_{k=0}^{p-1} s_k &= p^2 \sum_{k=0}^{p-1} \frac{1}{k+1} \binom{2k}{k} \left(2 \binom{p-1}{k} \binom{p-1}{k+1} - \binom{p-1}{k}^2 \right) \\ &\equiv p^2 \left(- \sum_{k=0}^{p-2} \frac{2}{k+1} \binom{2k}{k} - \sum_{k=0}^{p-1} \frac{1}{k+1} \binom{2k}{k} \right) \pmod{p^3}. \end{aligned}$$

The proof then follows from the congruences (2.10) and (2.11). \square

4 Proof of Theorem 1.3

Proof of (1.6). We need to prove the following identity:

$$\sum_{k=0}^{n-1} S_k^+ = n^2 \sum_{k=0}^{n-1} \left(\binom{2k+1}{k} \binom{n-1}{k}^2 + \frac{1}{k+1} \binom{2k}{k} \binom{n-1}{k} \binom{n-1}{k+1} \right). \quad (4.1)$$

Applying Zeilberger's algorithm, we find that the numbers S_n^+ satisfy the following recurrence:

$$\begin{aligned} &(n+3)^2(16n^2 + 69n + 71)S_{n+3}^+ - (176n^4 + 1911n^3 + 7600n^2 + 13101n + 8217)S_{n+2}^+ \\ &+ (304n^4 + 2943n^3 + 10721n^2 + 18171n + 12672)S_{n+1}^+ \\ &- 9(n+1)^2(16n^2 + 101n + 156)S_n^+ = 0. \end{aligned} \quad (4.2)$$

It follows by induction on n that

$$\begin{aligned} &(n+2)^2(256n^4 + 1280n^3 + 2176n^2 + 1488n + 363)S_{n+2}^+ \\ &- (2560n^6 + 25600n^5 + 100096n^4 + 194848n^3 + 198238n^2 + 99610n + 19677)S_{n+1}^+ \\ &+ 9(n+1)^2(256n^4 + 2304n^3 + 7552n^2 + 10704n + 5563)S_n^+ = 0, \end{aligned} \quad (4.3)$$

since we have

$$\begin{aligned} &(256n^4 + 2304n^3 + 7552n^2 + 10704n + 5563)\xi_n + (16n^2 + 69n + 71)\eta_{n+1} \\ &- (16n^2 + 101n + 156)\eta_n = 0, \end{aligned}$$

where ξ_n and η_n denote the left-hand sides of (4.2) and (4.3), respectively.

Denote the right-hand side of (4.1) by $n^2 e_{n-1}$, and let $y_n = (n+1)^2 e_n - n^2 e_{n-1}$. Applying Zeilberger's algorithm for y_n , we obtain

$$\begin{aligned} & (n+3)^2(1024n^5 + 9984n^4 + 36736n^3 + 64000n^2 + 53236n + 17057)y_{n+3} \\ & - (11264n^7 + 186624n^6 + 1275008n^5 + 4648064n^4 + 9750908n^3 + 11764759n^2 \\ & + 7570338n + 2011779)y_{n+2} + (19456n^7 + 324864n^6 + 2232448n^5 + 8173312n^4 \\ & + 17192092n^3 + 20752931n^2 + 13332438n + 3555639)y_{n+1} \\ & - 9(n+1)^2(1024n^5 + 15104n^4 + 86912n^3 + 244352n^2 + 336500n + 182037)y_n = 0. \end{aligned} \quad (4.4)$$

It follows that the recurrence (4.3) also holds for y_n , in view of

$$\begin{aligned} & (256n^4 + 2304n^3 + 7552n^2 + 10704n + 5563)\zeta_n \\ & + (1024n^5 + 9984n^4 + 36736n^3 + 64000n^2 + 53236n + 17057)\sigma_{n+1} \\ & - (1024n^5 + 15104n^4 + 86912n^3 + 244352n^2 + 336500n + 182037)\sigma_n = 0, \end{aligned}$$

where ζ_n and σ_n denote the left-hand sides of (4.4) and (4.3) (with S_n^+ replaced by y_n and so on), respectively. (4.1). Noticing that $S_n^+ = y_n$ for $n = 1, 2$ and $S_0^+ = 1$, we complete the proof. \square

Proof of (1.7). Letting $n = p$ be a prime in (4.1), we have

$$\begin{aligned} \sum_{k=0}^{p-1} S_k^+ &= p^2 \sum_{k=0}^{p-1} \left(\binom{2k+1}{k} \binom{p-1}{k}^2 + \frac{1}{k+1} \binom{2k}{k} \binom{p-1}{k} \binom{p-1}{k+1} \right) \\ &\equiv p^2 \left(\sum_{k=0}^{p-1} \frac{2k+1}{k+1} \binom{2k}{k} - \sum_{k=0}^{p-2} \frac{1}{k+1} \binom{2k}{k} \right) \pmod{p^3}. \end{aligned}$$

The proof then follows from the congruences (2.9)–(2.11). \square

5 Concluding remarks and open problems

It is worth mentioning that Z.-W. Sun [4, Conjecture 5.7] also gave an interesting q -analogue of (1.4). We hope that our proof of (1.4) will give some hints to tackle this q -analogue.

It seems that the congruences (1.2) and (1.6) can be further generalized as follows. Let

$$S_n^{(r)} = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} (2k+1)^r, \quad \text{and} \quad T_n^{(r)} = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} (2k+1)^r (-1)^k.$$

Numerical calculation suggests the following conjectures.

Conjecture 5.1 *Let n and r be positive integers and p a prime. Then*

$$\sum_{k=0}^{n-1} S_k^{(2r)} \equiv 0 \pmod{n^2},$$

$$\sum_{k=0}^{n-1} T_k^{(2r)} \equiv 0 \pmod{n^2}, \quad (5.1)$$

$$\sum_{k=0}^{p-1} T_k^{(2)} \equiv \frac{p^2}{2} \left(5 - 3 \left(\frac{p}{5} \right) \right) \pmod{p^3}. \quad (5.2)$$

Note that the $r = 1$ case of (5.1) was conjectured by Z.-W. Sun (see [4, Conjecture 5.6]).

Conjecture 5.2 *Let n and r be positive integers. Then there exist integers a_{2r-1} and b_r , independent of n , such that*

$$a_{2r-1} \sum_{k=0}^{n-1} S_k^{(2r-1)} \equiv 0 \pmod{n^2}, \quad (5.3)$$

$$b_r \sum_{k=0}^{n-1} k S_k^{(r)} \equiv 0 \pmod{n^2}. \quad (5.4)$$

It seems rather difficult (almost impossible) to find out the best possible values of a_{2r-1} and b_r for all r . Nevertheless, for small r , we propose the following conjecture.

Conjecture 5.3 *Let $a_3 = 3$, $a_5 = 15$, $a_7 = 21$, $a_9 = 15$, $a_{11} = 33$, $a_{13} = 1365$, and $a_{15} = 3$. Let $b_2 = 12$, $b_3 = 4$, $b_4 = 60$, $b_5 = 20$, $b_6 = 84$, $b_7 = 28$, $b_8 = 60$, $b_9 = 20$, $b_{10} = 132$, $b_{11} = 44$, and $b_{12} = 5460$. Then the congruences (5.3) and (5.4) hold.*

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